

# Dynamical systems gradient method for solving ill-conditioned linear algebraic systems

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## Abstract

A version of the Dynamical Systems Method (DSM) for solving ill-conditioned linear algebraic systems is studied in this paper. An *a priori* and *a posteriori* stopping rules are justified. An algorithm for computing the solution using a spectral decomposition of the left-hand side matrix is proposed. Numerical results show that when a spectral decomposition of the left-hand side matrix is available or not computationally expensive to obtain the new method can be considered as an alternative to the Variational Regularization.

**Keywords.** Ill-conditioned linear algebraic systems , Dynamical Systems Method (DSM), Variational Regularization

**MSC:** 65F10; 65F22

## 1 Introduction

The Dynamical Systems Method (DSM) was systematically introduced and investigated in [11] as a general method for solving operator equations, linear and nonlinear, especially ill-posed operator equations. In several recent publications various versions of the DSM, proposed in [11], were shown to be as efficient and economical as variational regularization methods. This was demonstrated, for example, for the problems of solving ill-conditioned linear algebraic systems [2], and stable numerical differentiation of noisy data [8], [9], [3].

The aim of this paper is to formulate a version of the DSM gradient method for solving ill-posed linear equations and to demonstrate numerical efficiency of this method. There is a large literature on iterative regularization methods. These methods can be derived from a suitable version of the DSM by a discretization (see [11]). In the Gauss-Newton-type version of the DSM one has to invert some linear operator, which is an expensive procedure. The same is true for regularized Newton-type versions of the DSM and of their iterative counterparts. In contrast, the DSM gradient method we study in this paper *does not* require inversion of operators.

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We want to solve equation

$$Au = f, \quad (1)$$

where  $A$  is a linear bounded operator in a Hilbert space  $H$ . We assume that (1) has a solution, possibly nonunique, and denote by  $y$  the unique minimal-norm solution to (1),  $y \perp \mathcal{N} := \mathcal{N}(A) := \{u : Au = 0\}$ ,  $Ay = f$ . We assume that the range of  $A$ ,  $R(A)$ , is not closed, so problem (1) is ill-posed. Let  $f_\delta$ ,  $\|f - f_\delta\| \leq \delta$ , be the noisy data. We want to construct a stable approximation of  $y$ , given  $\{\delta, f_\delta, A\}$ . There are many methods for doing this, see, e.g., [4], [5], [6], [11], [13], to mention a few books, where variational regularization, quasisolutions, quasinversion, iterative regularization, and the DSM are studied.

The DSM version we study in this paper consists of solving the Cauchy problem

$$\dot{u}(t) = -A^*(Au(t) - f), \quad u(0) = u_0, \quad u_0 \perp N, \quad \dot{u} := \frac{du}{dt}, \quad (2)$$

where  $A^*$  is the adjoint to operator  $A$ , and proving the existence of the limit  $\lim_{t \rightarrow \infty} u(t) = u(\infty)$ , and the relation  $u(\infty) = y$ , i.e.,

$$\lim_{t \rightarrow \infty} \|u(t) - y\| = 0. \quad (3)$$

If the noisy data  $f_\delta$  are given, then we solve the problem

$$\dot{u}_\delta(t) = -A^*(Au_\delta(t) - f_\delta), \quad u_\delta(0) = u_0, \quad (4)$$

and prove that, for a suitable stopping time  $t_\delta$ , and  $u_\delta := u_\delta(t_\delta)$ , one has

$$\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0. \quad (5)$$

In Section 2 these results are formulated precisely and recipes for choosing  $t_\delta$  are proposed.

The novel results in this paper include the proof of the discrepancy principle (Theorem 3), an efficient method for computing  $u_\delta(t_\delta)$  (Section 3), and an a priori stopping rule (Theorem 2).

Our presentation is essentially self-contained.

## 2 Results

Suppose  $A : H \rightarrow H$  is a linear bounded operator in a Hilbert space  $H$ . Assume that equation

$$Au = f \quad (6)$$

has a solution not necessarily unique. Denote by  $y$  the unique minimal-norm solution i.e.,  $y \perp \mathcal{N} := \mathcal{N}(A)$ . Consider the following Dynamical Systems Method (DSM)

$$\begin{aligned} \dot{u} &= -A^*(Au - f), \\ u(0) &= u_0, \end{aligned} \quad (7)$$

where  $u_0 \perp \mathcal{N}$  is arbitrary. Denote  $T := A^*A$ ,  $Q := AA^*$ . The unique solution to (7) is

$$u(t) = e^{-tT}u_0 + e^{-tT} \int_0^t e^{sT} ds A^* f.$$

Let us show that any ill-posed linear equation (6) with exact data can be solved by the DSM.

## 2.1 Exact data

**Theorem 1** *Suppose  $u_0 \perp \mathcal{N}$ . Then problem (7) has a unique solution defined on  $[0, \infty)$ , and  $u(\infty) = y$ , where  $u(\infty) = \lim_{t \rightarrow \infty} u(t)$ .*

**Proof.** Denote  $w := u(t) - y$ ,  $w_0 = w(0)$ . Note that  $w_0 \perp \mathcal{N}$ . One has

$$\dot{w} = -Tw, \quad T = A^*A. \quad (8)$$

The unique solution to (8) is  $w = e^{-tT}w_0$ . Thus,

$$\|w\|^2 = \int_0^{\|T\|} e^{-2t\lambda} d\langle E_\lambda w_0, w_0 \rangle.$$

where  $\langle u, v \rangle$  is the inner product in  $H$ , and  $E_\lambda$  is the resolution of the identity of the selfadjoint operator  $T$ . Thus,

$$\|w(\infty)\|^2 = \lim_{t \rightarrow \infty} \int_0^{\|T\|} e^{-2t\lambda} d\langle E_\lambda w_0, w_0 \rangle = \|P_{\mathcal{N}} w_0\|^2 = 0,$$

where  $P_{\mathcal{N}} = E_0 - E_{-0}$  is the orthogonal projector onto  $\mathcal{N}$ . Theorem 1 is proved.  $\square$

## 2.2 Noisy data $f_\delta$

Let us solve stably equation (6) assuming that  $f$  is not known, but  $f_\delta$ , the noisy data, are known, where  $\|f_\delta - f\| \leq \delta$ . Consider the following DSM

$$\dot{u}_\delta = -A^*(Au_\delta - f_\delta), \quad u_\delta(0) = u_0.$$

Denote

$$w_\delta := u_\delta - y, \quad T := A^*A, \quad w_\delta(0) = w_0 := u_0 - y \in \mathcal{N}^\perp.$$

Let us prove the following result:

**Theorem 2** *If  $\lim_{\delta \rightarrow 0} t_\delta = \infty$ ,  $\lim_{\delta \rightarrow 0} t_\delta \delta = 0$ , and  $w_0 \perp \mathcal{N}$ , then*

$$\lim_{\delta \rightarrow 0} \|w_\delta(t_\delta)\| = 0.$$

**Proof.** One has

$$\dot{w}_\delta = -Tw_\delta + \eta_\delta, \quad \eta_\delta = A^*(f_\delta - f), \quad \|\eta_\delta\| \leq \|A\|\delta. \quad (9)$$

The unique solution of equation (9) is

$$w_\delta(t) = e^{-tT}w_\delta(0) + \int_0^t e^{-(t-s)T}\eta_\delta ds.$$

Let us show that  $\lim_{t \rightarrow \infty} \|w_\delta(t)\| = 0$ . One has

$$\lim_{t \rightarrow \infty} \|w_\delta(t)\| \leq \lim_{t \rightarrow \infty} \|e^{-tT}w_\delta(0)\| + \lim_{t \rightarrow \infty} \left\| \int_0^t e^{-(t-s)T}\eta_\delta ds \right\|. \quad (10)$$

One uses the spectral theorem and gets:

$$\begin{aligned} \int_0^t e^{-(t-s)T} ds \eta_\delta &= \int_0^t \int_0^{\|T\|} dE_\lambda \eta_\delta e^{-(t-s)\lambda} ds \\ &= \int_0^{\|T\|} e^{-t\lambda} \frac{e^{t\lambda} - 1}{\lambda} dE_\lambda \eta_\delta = \int_0^{\|T\|} \frac{1 - e^{-t\lambda}}{\lambda} dE_\lambda \eta_\delta. \end{aligned} \quad (11)$$

Note that

$$0 \leq \frac{1 - e^{-t\lambda}}{\lambda} \leq t, \quad \forall \lambda > 0, t \geq 0, \quad (12)$$

since  $1 - x \leq e^{-x}$  for  $x \geq 0$ . From (11) and (12), one obtains

$$\begin{aligned} \left\| \int_0^t e^{-(t-s)T} ds \eta_\delta \right\|^2 &= \int_0^{\|T\|} \left| \frac{1 - e^{-t\lambda}}{\lambda} \right|^2 d\langle E_\lambda \eta_\delta, \eta_\delta \rangle \\ &\leq t^2 \int_0^{\|T\|} d\langle E_\lambda \eta_\delta, \eta_\delta \rangle \\ &= t^2 \|\eta_\delta\|^2. \end{aligned} \quad (13)$$

Since  $\|\eta_\delta\| \leq \|A\|\delta$ , from (10) and (13), one gets

$$\lim_{\delta \rightarrow 0} \|w_\delta(t_\delta)\| \leq \lim_{\delta \rightarrow 0} \left( \|e^{-t_\delta T}w_\delta(0)\| + t_\delta \delta \|A\| \right) = 0.$$

Here we have used the relation:

$$\lim_{\delta \rightarrow 0} \|e^{-t_\delta T}w_\delta(0)\| = \|P_{\mathcal{N}}w_0\| = 0,$$

and the last equality holds because  $w_0 \in \mathcal{N}^\perp$ . Theorem 2 is proved.  $\square$

From Theorem 2, it follows that the relation  $t_\delta = \frac{C}{\delta^\gamma}$ ,  $\gamma = \text{const}$ ,  $\gamma \in (0, 1)$  and  $C > 0$  is a constant, can be used as an *a priori* stopping rule, i.e., for such  $t_\delta$  one has

$$\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - y\| = 0. \quad (14)$$

### 2.3 Discrepancy principle

Let us consider equation (6) with noisy data  $f_\delta$ , and a DSM of the form

$$\dot{u}_\delta = -A^* A u_\delta + A^* f_\delta, \quad u_\delta(0) = u_0. \quad (15)$$

for solving this equation. Equation (15) has been used in Section 2.2. Recall that  $y$  denotes the minimal-norm solution of equation (6).

**Theorem 3** *Assume that  $\|Au_0 - f_\delta\| > C\delta$ . The solution  $t_\delta$  to the equation*

$$h(t) := \|Au_\delta(t) - f_\delta\| = C\delta, \quad C = \text{const}, \quad C \in (1, 2), \quad (16)$$

*does exist, is unique, and*

$$\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - y\| = 0. \quad (17)$$

**Proof.** Denote

$$v_\delta(t) := Au_\delta(t) - f_\delta, \quad T := A^* A, \quad Q = AA^*, \quad w(t) := u(t) - y, \quad w_0 := u_0 - y.$$

One has

$$\begin{aligned} \frac{d}{dt} \|v_\delta(t)\|^2 &= 2 \operatorname{Re} \langle A \dot{u}_\delta(t), Au_\delta(t) - f_\delta \rangle \\ &= 2 \operatorname{Re} \langle A[-A^*(Au_\delta(t) - f_\delta)], Au_\delta(t) - f_\delta \rangle \\ &= -2 \|A^* v_\delta(t)\|^2 \leq 0. \end{aligned} \quad (18)$$

Thus,  $\|v_\delta(t)\|$  is a nonincreasing function. Let us prove that equation (16) has a solution for  $C \in (1, 2)$ . Recall the known commutation formulas:

$$e^{-sT} A^* = A^* e^{-sQ}, \quad A e^{-sT} = e^{-sQ} A.$$

Using these formulas and the representation

$$u_\delta(t) = e^{-tT} u_0 + \int_0^t e^{-(t-s)T} A^* f_\delta ds,$$

one gets:

$$\begin{aligned} v_\delta(t) &= Au_\delta(t) - f_\delta \\ &= A e^{-tT} u_0 + A \int_0^t e^{-(t-s)T} A^* f_\delta ds - f_\delta \\ &= e^{-tQ} A u_0 + e^{-tQ} \int_0^t e^{sQ} ds Q f_\delta - f_\delta \\ &= e^{-tQ} A(u_0 - y) + e^{-tQ} f + e^{-tQ} (e^{tQ} - I) f_\delta - f_\delta \\ &= e^{-tQ} A w_0 - e^{-tQ} f_\delta + e^{-tQ} f. \end{aligned}$$

Note that

$$\lim_{t \rightarrow \infty} e^{-tQ} A w_0 = \lim_{t \rightarrow \infty} A e^{-tT} w_0 = A P_{\mathcal{N}} w_0 = 0.$$

Here the continuity of  $A$ , and the following relation

$$\lim_{t \rightarrow \infty} e^{-tT} w_0 = \lim_{t \rightarrow \infty} \int_0^{\|T\|} e^{-st} dE_s w_0 = (E_0 - E_{-0}) w_0 = P_{\mathcal{N}} w_0,$$

were used. Therefore,

$$\lim_{t \rightarrow \infty} \|v_\delta(t)\| = \lim_{t \rightarrow \infty} \|e^{-tQ}(f - f_\delta)\| \leq \|f - f_\delta\| \leq \delta, \quad (19)$$

because  $\|e^{-tQ}\| \leq 1$ . The function  $h(t)$  is continuous on  $[0, \infty)$ ,  $h(0) = \|Au_0 - f_\delta\| > C\delta$ ,  $h(\infty) \leq \delta$ . Thus, equation (16) must have a solution  $t_\delta$ .

Let us prove the uniqueness of  $t_\delta$ . Without loss of generality we can assume that there exists  $t_1 > t_\delta$  such that  $\|Au_\delta(t_1) - f_\delta\| = C\delta$ . Since  $\|v_\delta(t)\|$  is nonincreasing and  $\|v_\delta(t_\delta)\| = \|v_\delta(t_1)\|$ , one has

$$\|v_\delta(t)\| = \|v_\delta(t_\delta)\|, \quad \forall t \in [t_\delta, t_1].$$

Thus,

$$\frac{d}{dt} \|v_\delta(t)\|^2 = 0, \quad \forall t \in (t_\delta, t_1). \quad (20)$$

Using (18) and (20) one obtains

$$A^* v_\delta(t) = A^*(Au_\delta(t) - f_\delta) = 0, \quad \forall t \in [t_\delta, t_1].$$

This and (15) imply

$$\dot{u}_\delta(t) = 0, \quad \forall t \in (t_\delta, t_1). \quad (21)$$

One has

$$\begin{aligned} \dot{u}_\delta(t) &= -T u_\delta(t) + A^* f_\delta \\ &= -T \left( e^{-tT} u_0 + \int_0^t e^{-(t-s)T} A^* f_\delta ds \right) + A^* f_\delta \\ &= -T e^{-tT} u_0 - (I - e^{-tT}) A^* f_\delta + A^* f_\delta \\ &= -e^{-tT} (T u_0 - A^* f_\delta). \end{aligned} \quad (22)$$

From (22) and (21), one gets  $T u_0 - A^* f = e^{tT} e^{-tT} (T u_0 - A^* f) = 0$ . Note that the operator  $e^{tT}$  is an isomorphism for any fixed  $t$  since  $T$  is selfadjoint and bounded. Since  $T u_0 - A^* f = 0$ , by (22) one has  $\dot{u}_\delta(t) = 0$ ,  $u_\delta(t) = u_\delta(0)$ ,  $\forall t \geq 0$ . Consequently,

$$C\delta < \|Au_\delta(0) - f_\delta\| = \|Au_\delta(t_\delta) - f_\delta\| = C\delta.$$

This is a contradiction which proves the uniqueness of  $t_\delta$ .

Let us prove (17). First, we have the following estimate:

$$\begin{aligned}\|Au(t_\delta) - f\| &\leq \|Au(t_\delta) - Au_\delta(t_\delta)\| + \|Au_\delta(t_\delta) - f_\delta\| + \|f_\delta - f\| \\ &\leq \left\| e^{-t_\delta Q} \int_0^{t_\delta} e^{sQ} Q ds \right\| \|f_\delta - f\| + C\delta + \delta.\end{aligned}\tag{23}$$

Let us use the inequality:

$$\left\| e^{-t_\delta Q} \int_0^{t_\delta} e^{sQ} Q ds \right\| = \|I - e^{-t_\delta Q}\| \leq 2,$$

and conclude from (23), that

$$\lim_{\delta \rightarrow 0} \|Au(t_\delta) - f\| = 0.\tag{24}$$

Secondly, we claim that

$$\lim_{\delta \rightarrow 0} t_\delta = \infty.$$

Assume the contrary. Then there exist  $t_0 > 0$  and a sequence  $(t_{\delta_n})_{n=1}^\infty$ ,  $t_{\delta_n} < t_0$ , such that

$$\lim_{n \rightarrow \infty} \|Au(t_{\delta_n}) - f\| = 0.\tag{25}$$

Analogously to (18), one proves that

$$\frac{d\|v\|^2}{dt} \leq 0,$$

where  $v(t) := Au(t) - f$ . Thus,  $\|v(t)\|$  is nonincreasing. This and (25) imply the relation  $\|v(t_0)\| = \|Au(t_0) - f\| = 0$ . Thus,

$$0 = v(t_0) = e^{-t_0 Q} A(u_0 - y).$$

This implies  $A(u_0 - y) = e^{t_0 Q} e^{-t_0 Q} A(u_0 - y) = 0$ , so  $u_0 - y \in \mathcal{N}$ . Since  $u_0 - y \in \mathcal{N}^\perp$ , it follows that  $u_0 = y$ . This is a contradiction because

$$C\delta \leq \|Au_0 - f_\delta\| = \|f - f_\delta\| \leq \delta, \quad 1 < C < 2.$$

Thus,  $\lim_{\delta \rightarrow 0} t_\delta = \infty$ .

Let us continue the proof of (17). Let  $w_\delta(t) := u_\delta(t) - y$ . We claim that  $\|w_\delta(t)\|$  is nonincreasing on  $[0, t_\delta]$ . One has

$$\begin{aligned}\frac{d}{dt} \|w_\delta(t)\|^2 &= 2 \operatorname{Re} \langle \dot{u}_\delta(t), u_\delta(t) - y \rangle \\ &= 2 \operatorname{Re} \langle -A^*(Au_\delta(t) - f_\delta), u_\delta(t) - y \rangle \\ &= -2 \operatorname{Re} \langle Au_\delta(t) - f_\delta, Au_\delta(t) - f_\delta + f_\delta - Ay \rangle \\ &\leq -2 \|Au_\delta(t) - f_\delta\| \left( \|Au_\delta(t) - f_\delta\| - \|f_\delta - f\| \right) \\ &\leq 0.\end{aligned}$$

Here we have used the inequalities:

$$\|Au_\delta(t) - f_\delta\| \geq C\delta > \|f_\delta - Ay\| = \delta, \quad \forall t \in [0, t_\delta].$$

Let  $\epsilon > 0$  be arbitrary small. Since  $\lim_{t \rightarrow \infty} u(t) = y$ , there exists  $t_0 > 0$ , independent of  $\delta$ , such that

$$\|u(t_0) - y\| \leq \frac{\epsilon}{2}. \quad (26)$$

Since  $\lim_{\delta \rightarrow 0} t_\delta = \infty$ , there exists  $\delta_0$  such that  $t_\delta > t_0, \forall \delta \in (0, \delta_0)$ . Since  $\|w_\delta(t)\|$  is nonincreasing on  $[0, t_\delta]$  one has

$$\|w_\delta(t_\delta)\| \leq \|w_\delta(t_0)\| \leq \|u_\delta(t_0) - u(t_0)\| + \|u(t_0) - y\|, \quad \forall \delta \in (0, \delta_0). \quad (27)$$

Note that

$$\|u_\delta(t_0) - u(t_0)\| = \|e^{-t_0 T} \int_0^{t_0} e^{sT} ds A^* (f_\delta - f)\| \leq \|e^{-t_0 T} \int_0^{t_0} e^{sT} ds A^*\| \delta. \quad (28)$$

Since  $e^{-t_0 T} \int_0^{t_0} e^{sT} ds A^*$  is a bounded operator for any fixed  $t_0$ , one concludes from (28) that  $\lim_{\delta \rightarrow 0} \|u_\delta(t_0) - u(t_0)\| = 0$ . Hence, there exists  $\delta_1 \in (0, \delta_0)$  such that

$$\|u_\delta(t_0) - u(t_0)\| \leq \frac{\epsilon}{2}, \quad \forall \delta \in (0, \delta_1). \quad (29)$$

From (26)–(29), one obtains

$$\|u_\delta(t_\delta) - y\| = \|w_\delta(t_\delta)\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall \delta \in (0, \delta_1).$$

This means that  $\lim_{\delta \rightarrow 0} u_\delta(t_\delta) = y$ . Theorem 3 is proved.  $\square$

### 3 Computing $u_\delta(t_\delta)$

#### 3.1 Systems with known spectral decomposition

One way to solve the Cauchy problem (15) is to use explicit Euler or Runge-Kutta methods with a constant or adaptive stepsize  $h$ . However, stepsize  $h$  for solving (15) by explicit numerical methods is often smaller than 1 and the stopping time  $t_\delta = nh$  may be large. Therefore, the computation time, characterized by the number of iterations  $n$ , for this approach may be large. This fact is also reported in [2], where one of the most efficient numerical methods for solving ordinary differential equations (ODEs), the DOPRI45 (see [1]), is used for solving a Cauchy problem in a DSM. Indeed, the use of explicit Euler method leads to a Landweber iteration which is known for slow convergence. Thus, it may be computationally expensive to compute  $u_\delta(t_\delta)$  by numerical methods for ODEs.

However, when  $A$  in (15) is a matrix and a decomposition  $A = USV^*$ , where  $U$  and  $V$  are unitary matrices and  $S$  is a diagonal matrix, is known, it is possible to compute  $u_\delta(t_\delta)$  at a speed comparable to other methods such as the variational regularization (VR) as it will be shown below.



We have

$$u_\delta(t) = e^{-tT}u_0 + e^{-tT} \int_0^t e^{sT} ds A^* f_\delta, \quad T := A^* A. \quad (30)$$

Suppose that a decomposition

$$A = USV^*, \quad (31)$$

where  $U$  and  $V$  are unitary matrices and  $S$  is a diagonal matrix is known. These matrices possibly contain complex entries. Thus,  $T = A^* A = V \bar{S} S V^*$  and  $e^T = e^{V \bar{S} S V^*}$ . Using the formula  $e^{V \bar{S} S V^*} = V e^{\bar{S} S} V^*$ , which is valid if  $V$  is unitary and  $\bar{S} S$  is diagonal, equation (30) can be rewritten as

$$u_\delta(t) = V e^{-t \bar{S} S} V^* u_0 + V \int_0^t e^{(s-t) \bar{S} S} ds \bar{S} U^* f_\delta. \quad (32)$$

Here, the overbar stands for complex conjugation. Choose  $u_0 = 0$ . Then

$$u_\delta(t) = V \int_0^t e^{(s-t) \bar{S} S} ds \bar{S} h_\delta, \quad h_\delta := U^* f_\delta. \quad (33)$$

Let us assume that

$$A^* f_\delta \neq 0. \quad (34)$$

This is a natural assumption. Indeed, if  $A^* f_\delta = 0$ , then by the definition of  $h_\delta$  in (33), relation  $V^* V = I$ , and equation (31), one gets

$$\bar{S} h_\delta = \bar{S} U^* f_\delta = V^* V \bar{S} U^* f_\delta = V^* A^* f_\delta = 0. \quad (35)$$

Equations (35) and (33) imply  $u_\delta(t) \equiv 0$ .

The stopping time  $t_\delta$  we choose by the following discrepancy principle:

$$\|A u_\delta(t_\delta) - f_\delta\| = \left\| \int_0^{t_\delta} e^{(s-t_\delta) \bar{S} S} ds \bar{S} S h_\delta - h_\delta \right\| = \|e^{-t_\delta \bar{S} S} h_\delta\| = C \delta.$$

where  $1 < C < 2$ .

Let us find  $t_\delta$  from the equation

$$\phi(t) := \psi(t) - C \delta = 0, \quad \psi(t) := \|e^{-t \bar{S} S} h_\delta\|. \quad (36)$$

The existence and uniqueness of the solution  $t_\delta$  to equation (36) follow from Theorem 3.

We claim that *equation (36) can be solved by using Newton's iteration (43) for any initial value  $t_0$  such that  $\phi(t_0) > 0$ .*

Let us prove this claim. It is sufficient to prove that  $\phi(t)$  is a monotone strictly convex function. This is proved below.

Without loss of generality, we can assume that  $h_\delta$  (see (36)) is a vector with real components. The proof remained essentially the same for  $h_\delta$  with complex components.

First, we claim that

$$\sqrt{\bar{S} S} h_\delta \neq 0, \quad \text{and} \quad \|\sqrt{\bar{S} S} e^{-t \bar{S} S} h_\delta\| \neq 0, \quad (37)$$

so  $\psi(t) > 0$ .

Indeed, since  $e^{-t\bar{S}S}$  is an isomorphism and  $e^{-t\bar{S}S}$  commutes with  $\sqrt{\bar{S}S}$  one concludes that  $\|\sqrt{\bar{S}S}e^{-t\bar{S}S}h_\delta\| = 0$  iff  $\sqrt{\bar{S}S}h_\delta = 0$ . If  $\sqrt{\bar{S}S}h_\delta = 0$  then  $\bar{S}h_\delta = 0$ , and then, by equation (35),  $A^*f_\delta = \bar{S}h_\delta = 0$ . This contradicts to the assumption (34).

Let us now prove that  $\phi$  monotonically decays and is strictly convex. Then our claim will be proved.

One has

$$\frac{d}{dt}\langle e^{-t\bar{S}S}h_\delta, e^{-t\bar{S}S}h_\delta \rangle = -2\langle e^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle.$$

Thus,

$$\dot{\psi}(t) = \frac{d}{dt}\|e^{-t\bar{S}S}h_\delta\| = \frac{\frac{d}{dt}\|e^{-t\bar{S}S}h_\delta\|^2}{2\|e^{-t\bar{S}S}h_\delta\|} = -\frac{\langle e^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle}{\|e^{-t\bar{S}S}h_\delta\|}. \quad (38)$$

Equation (38), relation (37), and the fact that  $\langle e^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle = \|\sqrt{\bar{S}S}e^{-t\bar{S}S}h_\delta\|^2$  imply

$$\dot{\psi}(t) < 0. \quad (39)$$

From equation (38) and the definition of  $\psi$  in (36), one gets

$$\psi(t)\dot{\psi}(t) = -\langle e^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle \quad (40)$$

Differentiating equation (40) with respect to  $t$ , one obtains

$$\begin{aligned} \psi(t)\ddot{\psi}(t) + \dot{\psi}^2(t) &= \langle \bar{S}Se^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle + \langle e^{-t\bar{S}S}h_\delta, \bar{S}S\bar{S}Se^{-t\bar{S}S}h_\delta \rangle \\ &= 2\|\bar{S}Se^{-t\bar{S}S}h_\delta\|^2. \end{aligned}$$

This equation and equation (38) imply

$$\psi(t)\ddot{\psi}(t) = 2\|\bar{S}Se^{-t\bar{S}S}h_\delta\|^2 - \frac{\langle e^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle^2}{\|e^{-t\bar{S}S}h_\delta\|^2} \geq \|\bar{S}Se^{-t\bar{S}S}h_\delta\|^2 > 0. \quad (41)$$

Here the inequality:  $\langle e^{-t\bar{S}S}h_\delta, \bar{S}Se^{-t\bar{S}S}h_\delta \rangle \leq \|e^{-t\bar{S}S}h_\delta\|\|\bar{S}Se^{-t\bar{S}S}h_\delta\|$  was used. Since  $\psi > 0$ , inequality (41) implies

$$\ddot{\psi}(t) > 0. \quad (42)$$

It follows from inequalities (39) and (42) that  $\phi(t)$  is a strictly convex and decreasing function on  $(0, \infty)$ . Therefore,  $t_\delta$  can be found by Newton's iterations:

$$\begin{aligned} t_{n+1} &= t_n - \frac{\phi(t_n)}{\dot{\phi}(t_n)} \\ &= t_n + \frac{\|e^{-t_n\bar{S}S}h_\delta\| - C\delta}{\langle \bar{S}Se^{-t_n\bar{S}S}h_\delta, e^{-t_n\bar{S}S}h_\delta \rangle} \|e^{-t_n\bar{S}S}h_\delta\|, \quad n = 0, 1, \dots, \end{aligned} \quad (43)$$

for any initial guess  $t_0$  of  $t_\delta$  such that  $\phi(t_0) > 0$ . Once  $t_\delta$  is found, the solution  $u_\delta(t_\delta)$  is computed by (33).

**Remark 1** In the decomposition  $A = VSU^*$  we do not assume that  $U, V$  and  $S$  are matrices with real entries. The singular value decomposition (SVD) is a particular case of this decomposition.

It is computationally expensive to get the SVD of a matrix in general. However, there are many problems in which the decomposition (31) can be computed fast using the fast Fourier transform (FFT). Examples include image restoration problems with circulant block matrices (see [7]) and deconvolution problems. (see Section 4.2).

### 3.2 On the choice of $t_0$

Let us discuss a strategy for choosing the initial value  $t_0$  in Newton's iterations for finding  $t_\delta$ . We choose  $t_0$  satisfying condition:

$$0 < \phi(t_0) = \|e^{-t_0 \bar{S} S} h_\delta\| - \delta \leq \delta \quad (44)$$

by the following strategy

1. Choose  $t_0 := 10 \frac{\|h_\delta\|}{\delta}$  as an initial guess for  $t_0$ .
2. Compute  $\phi(t_0)$ . If  $t_0$  satisfying (44) we are done. Otherwise, we go to step 3.
3. If  $\phi(t_0) < 0$  and the inequality  $\phi(t_0) > \delta$  has not occurred in iteration, we replace  $t_0$  by  $\frac{t_0}{10}$  and go back to step 2. If  $\phi(t_0) < 0$  and the inequality  $\phi(t_0) > \delta$  has occurred in iteration, we replace  $t_0$  by  $\frac{t_0}{3}$  and go back to step 2. If  $\phi(t_0) > \delta$ , we go to step 4.
4. If  $\phi(t_0) > \delta$  and the inequality  $\phi(t_0) < 0$  has not occurred in iterations, we replace  $t_0$  by  $3t_0$  and go back to step 2. If the inequality  $\phi(t_0) < 0$  has occurred in some iteration before, we stop the iteration and use  $t_0$  as an initial guess in Newton's iterations for finding  $t_\delta$ .

## 4 Numerical experiments

In this section results of some numerical experiments with ill-conditioned linear algebraic systems are reported. In all the experiments, by DSMG we denote the version of the DSM described in this paper, by VR we denote the Variational Regularization, implemented using the discrepancy principle, and by DSM-[2] we denote the method developed in [2].

### 4.1 A linear algebraic system for the computation of second derivatives

Let us do some numerical experiments with linear algebraic systems arising in a numerical experiment of computing the second derivative of a noisy function.

The problem is reduced to an integral equation of the first kind. A linear algebraic system is obtained by a discretization of the integral equation whose kernel  $K$  is Green's function

$$K(s, t) = \begin{cases} s(t-1), & \text{if } s < t \\ t(s-1), & \text{if } s \geq t \end{cases}.$$

Here  $s, t \in [0, 1]$ . Using  $A_N$  from [2], we do some numerical experiments for solving  $u_N$  from the linear algebraic system  $A_N u_N = b_{N,\delta}$ . In the experiments the exact right-hand side is computed by the formula  $b_N = A_N u_N$  when  $u_N$  is given. In this test,  $u_N$  is computed by

$$u_N := (u(t_{N,1}), u(t_{N,2}), \dots, u(t_{N,N}))^T, \quad t_{N,i} := \frac{i}{N}, \quad i = 1, \dots, N,$$

where  $u(t)$  is a given function. We use  $N = 10, 20, \dots, 100$  and  $b_{N,\delta} = b_N + e_N$ , where  $e_N$  is a random vector whose coordinates are independent, normally distributed, with mean 0 and variance 1, and scaled so that  $\|e_N\| = \delta_{rel} \|b_N\|$ . This linear algebraic system is mildly ill-posed: the condition number of  $A_{100}$  is  $1.2158 \times 10^4$ .

In Figure 1, the difference between the exact solution and solution obtained by the DSMG, VR and DSM-[2] are plotted. In these experiments, we used  $N = 100$  and  $u(t) = \sin(\pi t)$  with  $\delta_{rel} = 0.05$  and  $\delta_{rel} = 0.01$ . Figure 1 shows that the results obtained by the VR and the DSM-[2] are very close to each other. The results obtained by the DSMG are much better than those by the DSM-[2] and by the VR.

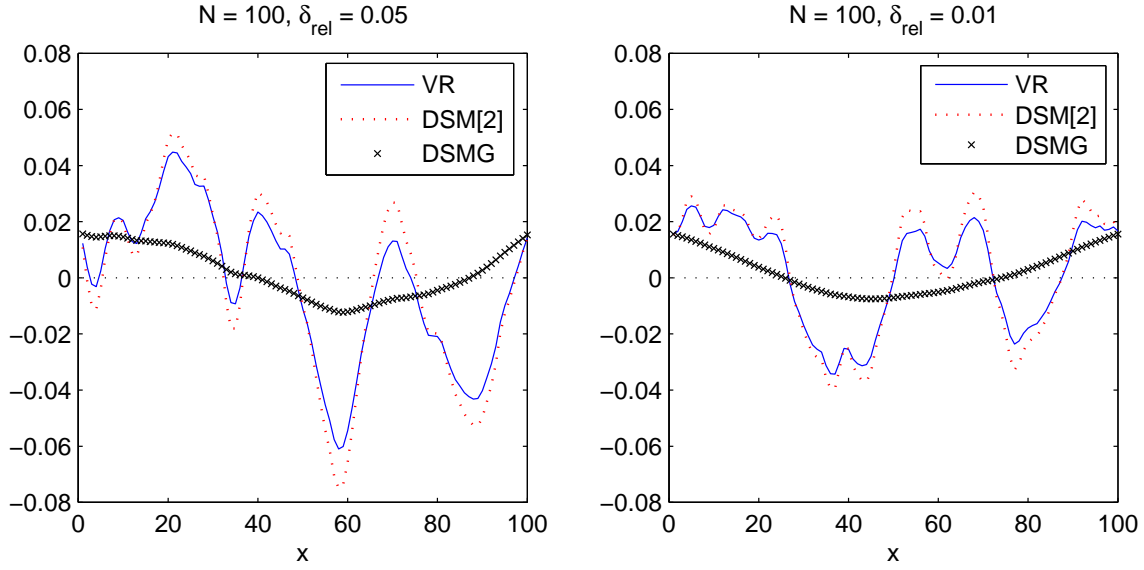


Figure 1: *Plots of differences between the exact solution and solutions obtained by the DSMG, VR and DSM-[2].*

Table 1 presents numerical results when  $N$  varies from 10 to 100,  $u(t) = \sin(2\pi t)$ , and  $t \in [0, 1]$ . In this experiment the DSMG yields more accurate solutions than the DSM-[2] and the VR. The DSMG in this experiment takes more iterations than the DSM-[2] and the VR to get a solution.

In this experiment the DSMG is implemented using the SVD of  $A$  obtained by the function *svd* in Matlab. As already mentioned, the SVD is a special case of the spectral decomposition (31). It is expensive to compute the SVD, in general. However, there are

Table 1: Numerical results for computing second derivatives with  $\delta_{rel} = 0.01$ .

$N$	DSM		DSM-[2]		VR	
	$n_{iter}$	$\frac{\ u_\delta - y\ _2}{\ y\ _2}$	$n_{insol}$	$\frac{\ u_\delta - y\ _2}{\ y\ _2}$	$n_{insol}$	$\frac{\ u_\delta - y\ _2}{\ y\ _2}$
20	9	0.0973	3	0.1130	6	0.1079
30	5	0.0831	4	0.1316	6	0.1160
40	7	0.0488	4	0.1150	6	0.1045
50	9	0.0614	4	0.1415	6	0.1063
60	6	0.0419	4	0.0919	6	0.0817
70	9	0.0513	4	0.0961	6	0.0842
80	6	0.0418	4	0.1225	6	0.0981
90	7	0.0287	4	0.0919	7	0.0840
100	7	0.0248	5	0.0778	7	0.0553

practically important problems where the spectral decomposition (31) can be computed fast (see Section 4.2 below). These problems consist of deconvolution problems using the Fast Fourier Transform (FFTs).

The conclusion from this experiment is: the DSMG may yield results with much better accuracy than the VR and DSM-[2]. Numerical experiments for various  $u(t)$  show that the DSMG competes favorably with the VR and the DSM-[2].

## 4.2 An application to image restoration

The image degradation process can be modeled by the following equation:

$$g_\delta = g + w, \quad g = h * f, \quad \|w\| \leq \delta, \quad (45)$$

where  $h$  represents a convolution function that models the blurring that many imaging systems introduce. For example, camera defocus, motion blur, imperfections of the lenses, all these phenomenon can be modeled by choosing a suitable  $h$ . The functions  $g_\delta$ ,  $f$ , and  $w$  are the observed image, the original signal, and the noise, respectively. The noise  $w$  can be due to the electronics used (thermal and shot noise), the recording medium (film grain), or the imaging process (photon noise).

In practice  $g$ ,  $h$  and  $f$  in equation (45) are often given as functions of a discrete argument and equation (45) can be written in this case as

$$g_{\delta,i} = g_i + w_i = \sum_{j=-\infty}^{\infty} f_j h_{i-j} + w_i, \quad i \in \mathbb{Z}. \quad (46)$$

Note that one (or both) signals  $f_j$  and  $h_j$  have compact support (finite length). Suppose that signal  $f$  is periodic with period  $N$ , i.e.,  $f_{i+N} = f_i$ , and  $h_j = 0$  for  $j < 0$  and  $j \geq N$ . Assume that  $f$  is represented by a sequence  $f_0, \dots, f_{N-1}$  and  $h$  is represented by  $h_0, \dots, h_{N-1}$ . Then the convolution  $h * f$  is periodic signal  $g$  with period  $N$ , and the elements of  $g$  are defined as

$$g_i = \sum_{j=0}^{N-1} h_j f_{(i-j) \bmod N}, \quad i = 0, 1, \dots, N-1. \quad (47)$$

Here  $(i-j) \bmod N$  is  $i-j$  modulo  $N$ . The discrete Fourier transform (DFT) of  $g$  is defined as the sequence

$$\hat{g}_k := \sum_{j=0}^{N-1} g_j e^{-i2\pi jk/N}, \quad k = 0, 1, \dots, N-1.$$

Denote  $\hat{g} = (\hat{g}_0, \dots, \hat{g}_{N-1})^T$ . Then equation (47) implies

$$\hat{g} = \hat{f}\hat{h}, \quad \hat{f}\hat{g} := (\hat{f}_0\hat{h}_0, \hat{f}_1\hat{h}_1, \dots, \hat{f}_{N-1}\hat{h}_{N-1})^T. \quad (48)$$

Let  $\text{diag}(a)$  denote a diagonal matrix whose diagonal is  $(a_0, \dots, a_{N-1})$  and other entries are zeros. Then equation (48) can be rewritten as

$$\hat{g} = A\hat{f}, \quad A := \text{diag}(\hat{h}). \quad (49)$$

Since  $A$  is of the form (31) with  $U = V = I$  and  $S = \text{diag}(\hat{h})$ , one can use the DSMG method to solve equation (49) stably for  $\hat{f}$ .

The image restoration test problem we use is taken from [7]. This test problem was developed at the US Air Force Phillips Laboratory, Lasers and Imaging Directorate, Kirtland Air Force Base, New Mexico. The original and blurred images have  $256 \times 256$  pixels, and are shown in Figure 2. These data has been widely used in the literature for testing image restoration algorithms.

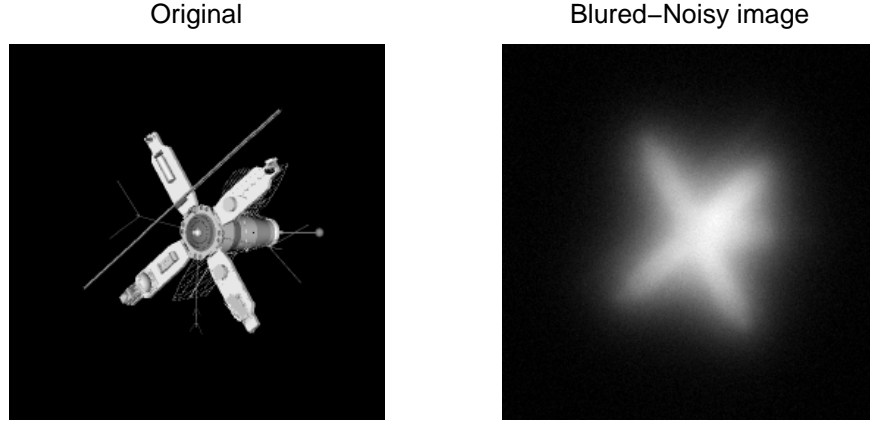


Figure 2: *Original and Blurred-noisy images.*

Figure 3 plots the regularized images by the VR and the DSMG when  $\delta_{rel} = 0.01$ . Again, with an input value for  $\delta_{rel}$ , the observed blurred-noisy images is computed by

$$g_\delta = g + \delta_{rel} \frac{\|g\|}{\|err\|} err,$$

where  $err$  is a vector with random entries normally distributed with mean 0 and variance 1. In this experiment, it took 5 and 8 iterations for the DSMG and the VR, respectively,

to yield numerical results. From Figure 3 one concludes that the DSMG is comparable to the VR in terms of accuracy. The time of computation in this experiment is about the same for the VR and DSMG.

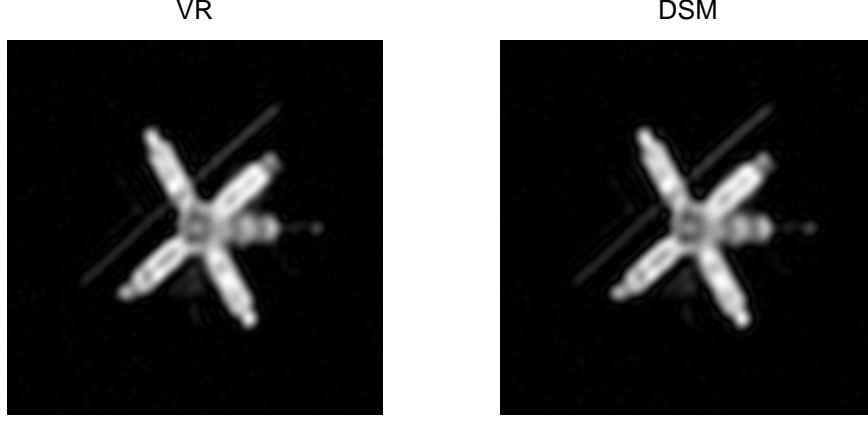


Figure 3: *Regularized images when noise level is 1%.*

Figure 4 plots the regularized images by the VR and the DSMG when  $\delta_{rel} = 0.05$ . It took 4 and 7 iterations for the DSMG and the VR, respectively, to yield numerical results. Figure 4 shows that the images obtained by the DSMG and the VR are about the same.

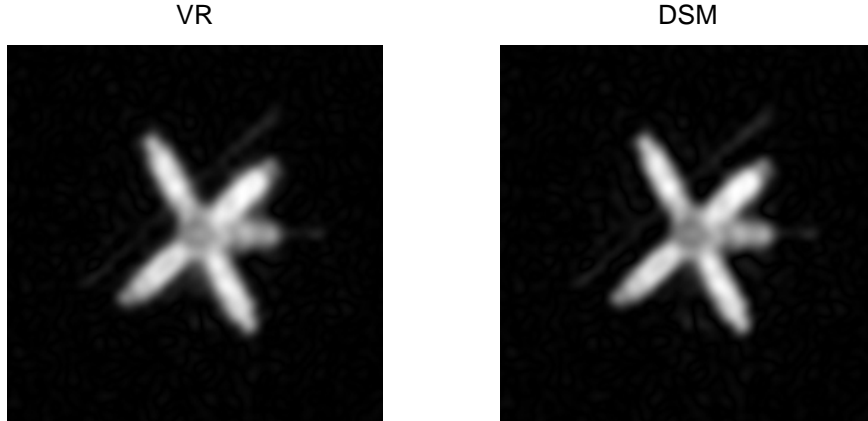


Figure 4: *Regularized images when noise level is 5%.*

The conclusions from this experiment are: the DSMG yields results with the same accuracy as the VR, and requires less iterations than the VR. The restored images by the DSM-[2] are about the same as those by the VR.

**Remark 2** Equation (45) can be reduced to equation (48) whenever one of the two functions  $f$  and  $h$  has compact support and the other is periodic.

## 5 Concluding remarks

A version of the Dynamical Systems Method for solving ill-conditioned linear algebraic systems is studied in this paper. An *a priori* and *a posteriori* stopping rules are formulated and justified. An algorithm for computing the solution in the case when a spectral decomposition of the matrix  $A$  is available is presented. Numerical results show that the DSMG, i.e., the DSM version developed in this paper, yields results comparable to those obtained by the VR and the DSM-[2] developed in [2], and the DSMG method may yield much more accurate results than the VR method. It is demonstrated in [7] that the rate of convergence of the Landweber method can be increased by using preconditioning techniques. The rate of convergence of the DSM version, presented in this paper, might be improved by a similar technique. The advantage of our method over the steepest descent in [7] is the following: *the stopping time  $t_\delta$  can be found from a discrepancy principle by Newton's iterations for a wide range of initial guess  $t_0$ ; when  $t_\delta$  is found one can compute the solution without any iterations.* Also, our method requires less iterations than the steepest descent in [7], which is an accelerated version of the Landweber method.

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